1 Problem 1.a

From the exercise we know that the dyadic Green's function is given by

$$\overleftrightarrow{G}(\mathbf{r},\mathbf{r}') = \left[\mathbb{1} + \frac{1}{k^2}\nabla\nabla\right], G_0(\mathbf{r},\mathbf{r}')$$

where the scalar Green's function $G_0(r, r')$ is given by

$$G_0(\mathbf{r},\mathbf{r}') = \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

Let's define

$$\frac{\partial^2}{\partial x \partial y} := \partial_{xy},$$

to write the components of the dyadic Green's function as

$$\overleftrightarrow{G}(\mathbf{r},\mathbf{r}') = \begin{pmatrix} k^2 + \partial_{xx} & \partial_{xy} & \partial_{xz} \\ \partial_{yx} & k^2 + \partial_{yy} & \partial_{yz} \\ \partial_{zx} & \partial_{zy} & k^2 + \partial_{zz} \end{pmatrix} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi k^2|\mathbf{r}-\mathbf{r}'|}.$$

Because $G_0(\mathbf{r},\mathbf{r}')$ only depends on $|\mathbf{r}-\mathbf{r}'|$, we define $R = |\mathbf{r}-\mathbf{r}'|$, $\mathbf{R} = \mathbf{r}-\mathbf{r}'$, and $\hat{\mathbf{R}} = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$. Hence, we obtain,

$$\nabla G_0(R) = \frac{\mathrm{d}}{\mathrm{d}R}\left(G_0(R)\right)\nabla R = \left(ik - \frac{1}{R}\right) G_0(R)\nabla R,$$

and with

$$\nabla R = \frac{x - x' + y - y' + z - z'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} = \frac{\mathbf{R}}{R} = \hat{\mathbf{R}},$$

it follows

$$\nabla G_0(R) = \left(ik - \frac{1}{R}\right) G_0(R)\hat{\mathbf{R}}.$$

Now we want to compute

$$\nabla\nabla G_0(R) = \nabla\left(ik - \frac{1}{R}\right) G_0(R)\hat{\mathbf{R}} + \left(ik - \frac{1}{R}\right)\nabla G_0(R)\hat{\mathbf{R}} + \left(ik - \frac{1}{R}\right) G_0(R)\nabla\hat{\mathbf{R}}.$$

At first, we consider the first term in the right-hand side of the equation,

$$\nabla\left(ik - \frac{1}{R}\right) = -\nabla\frac{1}{R} = -\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R}\right)\nabla R = \frac{\hat{\mathbf{R}}}{R^2}.$$

Now, we calculate $\nabla\hat{\mathbf{R}}$,

$$\nabla\hat{\mathbf{R}} = \nabla\left(\frac{\mathbf{R}}{R}\right) = \frac{\nabla\mathbf{R}}{R} + \mathbf{R}\nabla\frac{1}{R},$$

in which we can easily see that $\nabla\mathbf{R} = \mathbb{1}$ and $\nabla\frac{1}{R} = -\frac{\hat{\mathbf{R}}}{R^2}$. As a result,

$$\nabla\hat{\mathbf{R}} = (\mathbb{1} - \hat{\mathbf{R}}\hat{\mathbf{R}})\frac{1}{R}.$$

Now we just substitute all these parts in the first $\overleftrightarrow{G}(r, r')$ equation.

$$\begin{aligned}\overleftrightarrow{G}(R) &= \mathbb{1}G_0(R) + \frac{1}{k^2R^2}\hat{\mathbf{R}}\hat{\mathbf{R}}G_0(R) + \frac{1}{k^2}\left(ik - \frac{1}{R}\right)^2\hat{\mathbf{R}}\hat{\mathbf{R}}G_0 + \frac{1}{k^2}\left(ik - \frac{1}{R}\right)\frac{1}{R}(\mathbb{1} - \hat{\mathbf{R}}\hat{\mathbf{R}})G_0(R)\\ &= \left[\mathbb{1} + \frac{3}{k^2R^2}\hat{\mathbf{R}}\hat{\mathbf{R}} - \hat{\mathbf{R}}\hat{\mathbf{R}} - \frac{3i}{kR}\hat{\mathbf{R}}\hat{\mathbf{R}} - \frac{1}{k^2R^2}\mathbb{1} + \frac{i}{kR}\mathbb{1}\right]G_0(R).\end{aligned}$$

Finally we have

$$\overleftrightarrow{G}(R) = \left[(\mathbb{1} - \hat{\mathbf{R}}\hat{\mathbf{R}}) + \frac{i}{kR}(\mathbb{1} - 3\hat{\mathbf{R}}\hat{\mathbf{R}}) - \frac{1}{k^2R^2}(\mathbb{1} - 3\hat{\mathbf{R}}\hat{\mathbf{R}})\right]G_0(R).$$

The first term in the bracket $(\mathbb{1} - \hat{\mathbf{R}}\hat{\mathbf{R}})$ determines the far field and the rest expresses the near field.

2 Problem 1.b

The Fourier spectrum of the electric field, as it evolves along the z-axis, can be written as

$$E(k_x, k_y, z) = E(k_x, k_y, 0)e^{\pm ik_z z},$$

where the inverse Fourier transform is given by

$$E(x, y, z) = \iint \mathrm{d}k_x \mathrm{d}k_y\ E(k_x, k_y, z)e^{ik_x x}e^{ik_y y}.$$

We have to derive an equation by considering the evolution of the Fourier component by inserting the inverse Fourier transform in the vector Helmholtz equation given below,

$$(\nabla^2 + k^2)E(x, y, z) = 0,$$

where, $\nabla^2 = (\partial_x^2 + \partial_y^2 + \partial_z^2)$.

Since the inverse Fourier transform is a linear expression, we can exchange the integration with the derivative. Therefore, we have the Helmholtz equation written on the Fourier components of the electric field,

$$\iint \mathrm{d}k_x \mathrm{d}k_y(\nabla^2 + k^2)E(k_x, k_y, z)e^{i(k_x x + k_y y)} = 0.$$

Then we have,

$$\iint \mathrm{d}k_x \mathrm{d}k_y k_z^2 E(k_x, k_y, z)e^{i(k_x x + k_y y)} + \partial_z^2 E(k_x, k_y, z)e^{i(k_x x + k_y y)} = 0,$$

where $k_z^2 = (-k_x^2 - k_y^2 + k^2)$. Hence, the condition becomes, $[\partial_z^2 + k_z^2]E(k_x, k_y, z) = 0$. This is a 1D equation, with solution

$$E(k_x, k_y, z) = E(k_x, k_y, 0)\ e^{\pm ik_z z}.$$

3 Problem 2

$$\mathbf{A}(x,y,z-z_0) = \frac{-ikZ_{\mu\epsilon}}{4\pi}\frac{e^{ik\sqrt{x^2+y^2+(z-z_0)^2}}}{\sqrt{x^2+y^2+(z-z_0)^2}}\mathbf{n}_x$$
$$= \frac{-ikZ_{\mu\epsilon}}{4\pi}\frac{i}{2\pi}\mathbf{n}_x\iint\limits_{-\infty}^{\infty}\frac{e^{ik_xx+ik_yy+ik_z|z-z_0|}}{k_z}\mathrm{d}k_x\mathrm{d}k_y.$$

For the field we have

$$\mathbf{E}(x,y,z-z_0) = i\omega\left(1+\frac{1}{k^2}\nabla\nabla\cdot\right)\mathbf{A}(x,y,z-z_0).$$

As we already know,

$$\nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) \qquad , \qquad \nabla\cdot\mathbf{n}_x = \frac{\partial}{\partial x}$$

Now, with the substitution of $\mathbf{A}(x,y,z-z_0)$ in the field equation, one obtains

$$\mathbf{E} = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2}\left[(\mathbb{1}).\mathbf{n}_x\iint\limits_{-\infty}^{\infty}\frac{e^{ik_xx+ik_yy+ik_z|z-z_0|}}{k_z}\,\mathrm{d}k_x\,\mathrm{d}k_y + \frac{1}{k^2}\nabla\iint\limits_{-\infty}^{\infty}\frac{ik_x}{k_z}e^{ik_xx+ik_yy+ik_z|z-z_0|}\,\mathrm{d}k_x\,\mathrm{d}k_y\right]$$
$$= \frac{i\omega Z_{\mu\epsilon}}{8\pi^2}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}\iint\limits_{-\infty}^{\infty}\frac{e^{ik_xx+ik_yy+ik_z|z-z_0|}}{k_z}\,\mathrm{d}k_x\,\mathrm{d}k_y + \frac{1}{k^2}\iint\limits_{-\infty}^{\infty}\frac{ik_x}{k_z}\begin{pmatrix}ik_x\\ik_y\\ik_z\,sgn(z-z_0)\end{pmatrix}e^{ik_xx+ik_yy+ik_z|z-z_0|}\,\mathrm{d}k_x\,\mathrm{d}k_y\right]$$

As we defined before the angular spectrum representation of the electric field,

$$\mathbf{E}(x,y;z) = \iint\limits_{-\infty}^{\infty}\hat{\mathbf{E}}(k_x,k_y,z)e^{i(k_xx+k_yy)}\,\mathrm{d}k_x\,\mathrm{d}k_y,$$

a comparison of these two equations gives us

$$\hat{\mathbf{E}}(k_x,k_y,0) = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2}\frac{e^{ik_z|z-z_0|}}{k_z}\begin{pmatrix}1-k_x^2/k^2\\-k_xk_y/k^2\\-(k_xk_z/k^2)\,sgn(z-z_0)\end{pmatrix}.$$

For $z\to+\infty$ the evanescent waves will disappear, which means that only the Fourier terms with $k^2\geq k_x^2+k_y^2$ must be considered, hence

$$\lim_{z\to+\infty}\hat{\mathbf{E}}(k_x,k_y,0) = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2}\frac{e^{ik_zz}}{k_z}\begin{pmatrix}1-k_x^2/k^2\\-k_xk_y/k^2\\-(k_xk_z/k^2)\end{pmatrix},$$

and for the electric field we have

$$\mathbf{E}(x,y,z) = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2}\iint\limits_{k^2\geq k_x^2+k_y^2}\begin{pmatrix}1-k_x^2/k^2\\-k_xk_y/k^2\\-(k_xk_z/k^2)\end{pmatrix}\frac{e^{ik_xx+ik_yy+ik_zz}}{k_z}\,\mathrm{d}k_x\,\mathrm{d}k_y.$$

Now we change the coordinate system,

$$\begin{aligned} k_x &= k \sin\theta \cos\phi, \\ k_y &= k \sin\theta \sin\phi, \\ k_z &= k \cos\theta, \\ \mathrm{d}k_x \mathrm{d}k_y &= k^2 \sin\theta \mathrm{d}\theta \mathrm{d}\phi. \end{aligned}$$

and rewrite the equations as

$$\lim_{z\to+\infty} \hat{\mathbf{E}}(k_x, k_y, 0) = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2} \frac{e^{ik\cos\theta(z-z_0)}}{k\cos\theta} \begin{pmatrix} 1-\sin^2\theta\cos^2\phi \\ -\sin^2\theta\cos\phi\sin\phi \\ -\sin\theta\cos\theta\cos\phi \end{pmatrix},$$

$$\mathbf{E}(x,y,z) = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2 k} \iint\limits_{k^2\geq k_x^2+k_y^2} \begin{pmatrix} 1-\sin^2\theta\cos^2\phi \\ -\sin^2\theta\cos\phi\sin\phi \\ -\sin\theta\cos\theta\cos\phi \end{pmatrix} \frac{e^{ik[\sin\theta\cos\phi x+\sin\theta\sin\phi y+\cos\theta(z-z_0)]}}{\cos\theta} \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi.$$

For simplicity, we consider the case, $x = y = 0$,

$$\mathbf{E}(0,0,z) = \frac{i\omega Z_{\mu\epsilon}}{8\pi^2 k} \iint\limits_{k^2\geq k_x^2+k_y^2} \begin{pmatrix} 1-\sin^2\theta\cos^2\phi \\ -\sin^2\theta\cos\phi\sin\phi \\ -\sin\theta\cos\theta\cos\phi \end{pmatrix} \frac{e^{ik\cos\theta(z-z_0)]}}{\cos\theta} \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi$$

We know that

$$\begin{aligned} &\int_0^{2\pi} \cos\phi \,\mathrm{d}\phi = 0, \\ &\int_0^{2\pi} \cos\phi\sin\phi \,\mathrm{d}\phi = 0, \\ &\int_0^{2\pi} \cos^2\phi \,\mathrm{d}\phi = \int_0^{2\pi} (1-\sin^2\phi) \,\mathrm{d}\phi = \pi. \end{aligned}$$

Then only the x-component of the electric field is not zero, which reads

$$\begin{aligned} E_x(0,0,z) &= \frac{i\omega Z_{\mu\epsilon}}{8\pi^2 k} \int_0^\pi \pi(2-\sin^2\theta) \frac{e^{ik(z-z_0)\cos\theta}}{\cos\theta} \sin\theta \,\mathrm{d}\theta, \\ E_x(0,0,z) &= \frac{i\omega Z_{\mu\epsilon}}{8\pi k} \int_0^\pi \frac{(1+\cos^2\theta)}{\cos\theta} e^{ik(z-z_0)\cos\theta} \sin\theta \,\mathrm{d}\theta, \\ E_x(0,0,z) &= \frac{i\omega Z_{\mu\epsilon}}{8\pi k} \int_{-1}^1 \frac{1+x^2}{x} e^{ik(z-z_0)x} \,\mathrm{d}x. \end{aligned}$$