## Problem 1

We are looking for Fourier transform of

$$f(t)=\begin{cases}\cos^2\omega_p t & |t|<T\ \ :-T<t<T\\ 0 & |t|>T: t>T\ \ and\ \ t<-T\end{cases}$$

$$F(\omega)=\int_{-\infty}^{+\infty} f(t)\,e^{i\omega t}dt$$

$$=\int_{-\infty}^{-T} 0dt+\int_{-T}^{T}\cos^2(\omega_p t)\,e^{i\omega t}dt+\int_{T}^{+\infty}0dt$$

$$=\int_{-T}^{+T}\left[\tfrac{1}{2}+\tfrac{1}{2}\cos(2\omega_p t)\right]e^{i\omega t}dt$$

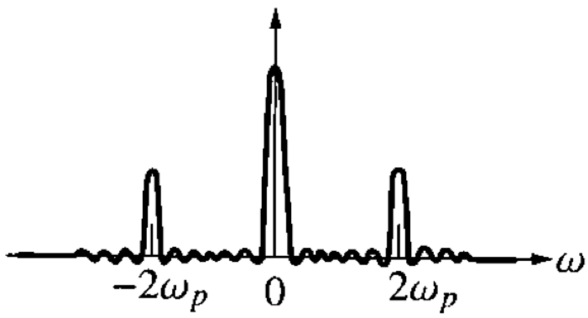
$$=\int_{-T}^{+T}\tfrac{1}{2}e^{i\omega t}dt+\tfrac{1}{2}\int_{-T}^{+T}\cos(2\omega_p t)e^{i\omega t}dt$$

$$=\tfrac{1}{2i\omega}\left(e^{i\omega T}-e^{-i\omega T}\right)+\tfrac{1}{4}\int_{-T}^{+T}\left(e^{2i\omega_p t}+e^{-2i\omega_p t}\right)e^{i\omega t}dt$$

$$=\tfrac{1}{\omega}\sin(\omega T)+\tfrac{1}{4}\tfrac{1}{i(2\omega_p+\omega)}e^{i(2\omega_p t+\omega t)}|_{-T}^{T}+\tfrac{1}{4}\tfrac{1}{-i(2\omega_p-\omega)}e^{-i(2\omega_p t-\omega t)}|_{-T}^{T}$$

$$F(\omega)=\frac{1}{\omega}\sin(\omega T)+\frac{1}{4\omega_p+2\omega}\sin(2\omega_p T+\omega T)-\frac{1}{4\omega_p-2\omega}\sin(\omega T-2\omega_p T)$$

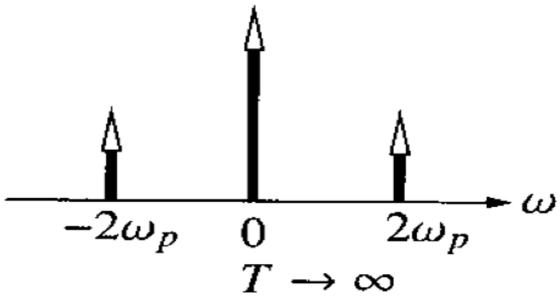
Because the $sinc(\omega)$ function is maximum in $\omega=0$




$$F(\omega)=\int_{-T}^{T}\tfrac{1}{2}e^{i\omega t}dt+\tfrac{1}{2}\int_{-T}^{T}\cos(2\omega_p t)e^{i\omega t}dt$$
$$=\int_{-T}^{T}e^{i\omega t}dt+\tfrac{1}{4}\int_{-T}^{T}\left(e^{2i\omega_p t}+e^{-2i\omega_p t}\right)e^{i\omega t}dt$$

but

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i\omega t}dt = \delta(\omega)$$
$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} e^{i(\omega\mp 2\omega_p)t}dt = \delta(\omega \mp 2\omega_p)$$




## Problem 2

$$F(k) = Ff(x)$$

a) $F\{1\} = 2\pi\delta(k)$

The inverse Fourier transform of $F(k)$ is defined as

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} F(k)\, e^{ikx}dk$$

On the other hand $F^{-1}\{F(k)\} = f(x)$ and according to the question $f(x) = 1$ and $F(k) = 2\pi\delta(k)$.

Now if we put $F(k) = 2\pi\delta(k)$ in Fourier inverse

$$f(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} 2\pi\delta(k)\, e^{ikx}dk$$

$$for \delta the integral is \quad \int\limits_{-\infty}^{\infty} f(x)\delta(x - x_0)\, dx = f(x_0)$$
$$\to \quad f(x) = e^{(0)} = 1$$

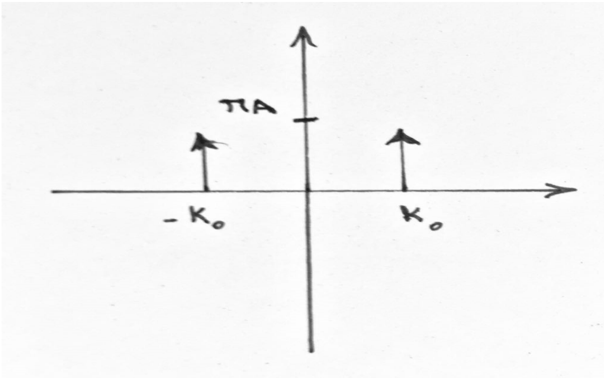
b) $f(x) = A\cos k_0 x$

$$F(f(x)) = \int\limits_{-\infty}^{\infty} A\cos k_0 x e^{ikx}dx = \int\limits_{-\infty}^{\infty} \tfrac{A}{2}\left(e^{ik_0x} + e^{-ik_0x}\right)e^{ikx}dx$$
$$= F(f(x)) = \tfrac{A}{2}\int\limits_{-\infty}^{\infty}\left(e^{i(k_0+k)x} + e^{i(k_0-k)x}\right)dx$$

as we know the delta diraac is

$$2\pi\delta(k) = \int_{-\infty}^{\infty} e^{ikx} dx$$

$$\rightarrow F\left(f\left(x\right)\right) = \pi A\left(\delta\left(k_0 + k\right) + \delta\left(k_0 - k\right)\right)$$




## Problem 3

$$rect\left|\frac{x-x_0}{a}\right| = \begin{cases} 0 & \left|\left(x-x_0\right)/a\right| > \frac{1}{2} \quad \rightarrow \quad x > \frac{a}{2} + x_0 \;\; and \;\; x < -\frac{a}{2} + x_0 \\ \frac{1}{2} & \left|\left(x-x_0\right)/a\right| \;\; = \frac{1}{2} \\ 1 & \left|\left(x-x_0\right)/a\right| < \frac{1}{2} \quad \rightarrow \quad -\frac{a}{2} + x_0 < x < \frac{a}{2} + x_0 \end{cases}$$

$$F\left(k\right) = \int_{-\infty}^{-\frac{a}{2}} 0e^{ikx}dx + \int_{-\frac{a}{2}}^{\frac{a}{2}} 1e^{ikx}dx + \int_{\frac{a}{2}}^{\infty} 0e^{ikx}dx$$

$$= \frac{e^{ikx}}{ik}\Big|_{-\frac{a}{2}}^{\frac{a}{2}} = \frac{1}{ik}\left[e^{ik\left(\frac{a}{2}\right)} - e^{ik\left(-\frac{a}{2}\right)}\right]$$

$$= \frac{2i}{ik}\sin\left(k\frac{a}{2}\right)$$

$$\rightarrow F\left(k\right) = \frac{2}{k}\sin\left(k\frac{a}{2}\right)$$

## Problem 4

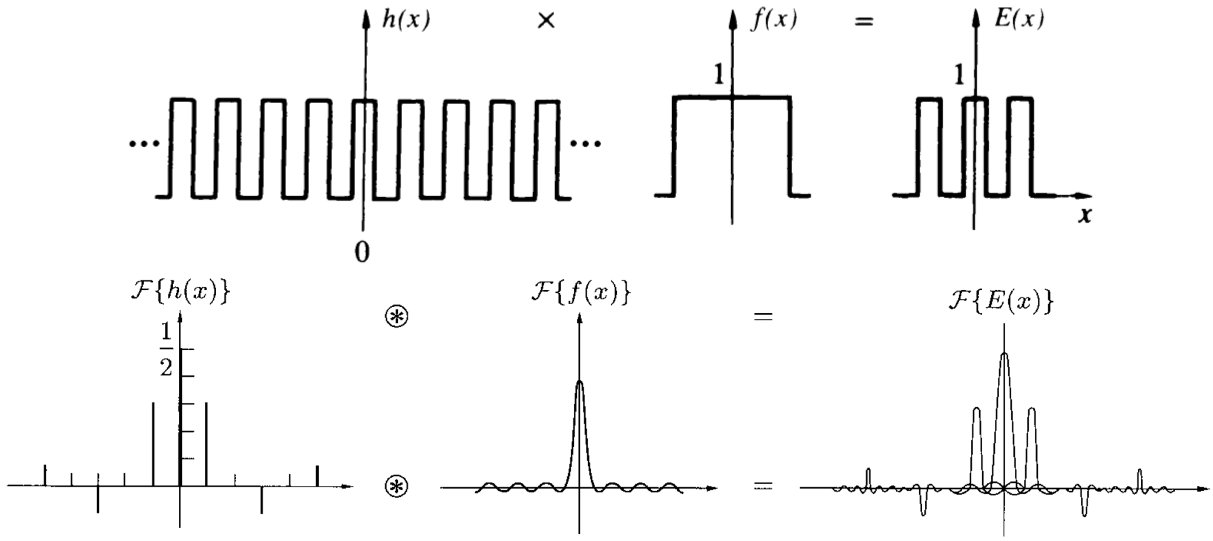
Any periodic signal $x(t)$, can be reconstructed from sine and cosine waves with frequencies that are multiplies of the fundamental $\omega$. The $a_n$ and $b_n$ coefficients hold the amplitudes of the cosine and sine waves, respectively.

$$x(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\omega t n) - \sum_{n=1}^{\infty} b_n \sin(\omega t n)$$

The The Fourier series analysis equations are:

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\mathrm{d}t,$$
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos(\omega tn)\mathrm{d}t,$$
$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(\omega tn)\mathrm{d}t,$$

and in this problem we have:




## Problem 5

$f(t) = A\sin(\omega t + \varepsilon)$

$$C_{ff}(\tau) = \lim_{T\to\infty} \tfrac{1}{2T} \int\limits_{-T}^{T} f(t)\, f(t-\tau)\, dt$$

$$= \lim_{T\to\infty} \tfrac{1}{2T} \int\limits_{-T}^{T} A\sin(\omega t + \varepsilon)\, A\sin(\omega t - \omega\tau + \varepsilon)\, dt$$

we also know that
$$\frac{1}{2}\cos\alpha - \frac{1}{2}\cos\beta = -\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

then $\alpha = \omega\tau$ and $\beta = 2\omega t - \omega\tau + 2\varepsilon$ then

$$C_{ff} = \lim_{T\to\infty} \frac{A^2}{2T} \int\limits_{-T}^{T} \left[\tfrac{1}{2}\cos(\omega\tau) - \tfrac{1}{2}\cos(2\omega t - \omega\tau + 2\varepsilon)\right] dt$$

$$= \lim_{T\to\infty} \frac{A^2}{2T} \left[\tfrac{1}{2}\cos(\omega\tau)(2T) - \tfrac{1}{2}\overbrace{\frac{\sin(2\omega T - \omega\tau + 2\varepsilon) - \sin(-2\omega T - \omega\tau + 2\varepsilon)}{2\omega}}^{=0}\right]$$

$$\to C_{ff} = \frac{A^2}{2}\cos(\omega\tau)$$