## Qubits and Quantum gates

1. Show that the Pauli matrix $\sigma_x$ (it is also called X gate) is the quantum analog of the NOT gate.
2. Consider a two-level system (qubit of the NV color center) defined by $\alpha |0\rangle + \beta |1\rangle$, show that the operation of X gate on the qubit exchange the probability of finding the qubit in the $|0\rangle$ and $|1\rangle$ state.
3. Show that the Z gate does not change the state of the qubit.
4. Show that the H gate brings the qubit state from $|0\rangle$ (or $|1\rangle$) state to superposition states. What is the probability of finding $|0\rangle$ and $|1\rangle$ states after the application of H gate.

## Quantum states readout

We know that the no-cloning theorem forbids us from copying arbitrary quantum states,

1. Show that if there exists a unitary U taking $|\psi\rangle \to |\phi\rangle$, there must exist another unitary V independent of $|\psi\rangle$ and $|\phi\rangle$. In another words, no information is lost when applying a unitary to a quantum state.
2. Suppose Alice holds a qubit in the state $|\psi\rangle = a|0\rangle + b|1\rangle$ and wants Bob to have that state as well. Why doesn't the following work? Alice measures her qubit in some basis of her choice, then prepares another qubit in the state she obtains and sends that to Bob. (Please answer without making explicit use of the no-cloning theorem.)

   We now introduce a scheme by which Alice can prepare $|\psi\rangle$ on Bob's side without sending him her qubit in fact, without sending any quantum information at all!, provided they share a Bell state. To be precise, the initial setup is as follows: Alice holds $|\psi\rangle_s$ and Alice and Bob each have a qubit of:

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \tag{1}$$

   So that the joint state of all three qubits is:

$$|\psi\rangle_{SAB} = |\psi\rangle_S \otimes |\phi^+\rangle_{AB} = \Big(a|0\rangle_S + b|1\rangle_S\Big) \otimes \frac{1}{\sqrt{2}}\Big(|00\rangle_{AB} + |11\rangle_{AB}\Big) \tag{2}$$

3. As with a single qubit, any state of a two-qubit system can be written in terms of an orthonormal basis, and also measured in such a basis. One example is the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Find a state $|\psi^-\rangle$ that, together with the following three states:

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB})$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB})$$

$\psi^-$ forms an orthonormal basis of the two qubit space. This basis is called the Bell states.

4. Rewrite the joint state (2) as a linear combination of the form $\sum_{i=1}^{4} = |\alpha_i\rangle_{SA} |\beta_i\rangle_B$, where $|\alpha\rangle$ ranges over the four possible Bell states on Alice's two qubits S and A, and $|\beta\rangle$ is a single qubit state on Bob's qubit.

5. Suppose Alice measures her two qubits SA in the Bell basis and sends the results to Bob. Show that for each of the four possible outcomes, Bob can use this (classical!) information to determine a unitary, independent of $|\psi\rangle_S$, on his qubit that will map it, in all cases, to the original state $|\psi\rangle_B$ that Alice had.