## Problem 1

Many Processes involve the absorption of single photons from quantum field state, the process of absorption being represented by the action of the annihilation operator $\hat{a}$. For an arbitrary field state $|\psi\rangle$, the absorption of a single photon yields the state $|\psi'\rangle \approx \hat{a}\,|\psi\rangle$. Normalize this state. Compare the average photon numbers $\bar{n}$ of the state $|\psi\rangle$ and $\bar{n}'$ of $|\psi'\rangle$. Do you find that $\bar{n}' = \bar{n} - 1$ ?

**For an arbitrary field state $|\psi\rangle$, the absorption of a single-photon yield the state as:**

$$\left|\psi'\right\rangle \approx \hat{a}\,|\psi\rangle$$

or

$$\left\langle\psi'\right| \approx \langle\psi|\,\hat{a}^{\dagger}$$

Normalization of this state:

$$\left|\psi'\right\rangle = C\,|\psi\rangle$$

$$\left\langle\psi'\middle|\psi'\right\rangle = |C|^2\,\langle\psi|\,\hat{a}^{\dagger}\hat{a}\,|\psi\rangle = 1$$

but

$$\langle\psi|\,\hat{a}^{\dagger}\hat{a}\,|\psi\rangle = \bar{n}$$

where $\bar{n}$ is the mean photon number of the state $|\psi\rangle$. Hence:

$$\langle\psi|\psi\rangle = |C|^2\bar{n} = 1 \implies C = \frac{1}{\sqrt{\bar{n}}}$$

Now one can write the new state as:

$$\left|\psi'\right\rangle = \frac{1}{\sqrt{\bar{n}}}\,|\psi\rangle$$

**The mean photon number of this state is given as:**

$$\bar{n'} = \left\langle\psi'\right|\hat{n}\left|\psi'\right\rangle = \frac{1}{\bar{n}}\,\langle\psi|\,\hat{a}^{\dagger}\hat{a}^{\dagger}\hat{a}\hat{a}\,|\psi\rangle$$

Remind that $[a, \hat{a}^{\dagger}] = 1$ and using $\hat{n} = \hat{a}^{\dagger}\hat{a}$, one can write :

$$\bar{n'} = \frac{1}{\bar{n}}\Big[\,\langle\psi|\,\hat{n}^2\,|\psi\rangle - \bar{n}\Big]$$

For a pure state (Fock state)$|n\rangle$,$< \hat{n^2} >=< \hat{n}^2 >= \bar{n}^2$, hence $\bar{n'} = \bar{n} - 1$, but for the other states $\bar{n'} \neq \bar{n} - 1$.

## Problem 2

Consider the superposition of the vacuum and 10 photon number state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big(\,|0\rangle + |10\rangle\,\Big)$$

Calculate the average photon number for this state. Next, assume that a single photon is absorbed and recalculate the average photon number. Does your result seem sensible in comparison with your answer to the previous question ?

**The quantum state is the superposition of the vacuum state and the 10 photon number state**. Thus:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\Big( |0\rangle + |10\rangle \Big)$$

The average photon number $\bar{n}$ for this state is given as:

$$\bar{n} =< \hat{n} >= \langle\psi| \hat{a}^\dagger\hat{a} |\psi\rangle$$

$$\bar{n} = \frac{1}{2}\Big[ \langle 0| \hat{a}^\dagger\hat{a} |0\rangle + \langle 10| \hat{a}^\dagger\hat{a} |10\rangle \Big] = \frac{1}{2} \times 10 = 5 \implies \bar{n} = 5$$

**If we assume a single photon is absorber, our normalized state can be transformed as:** $|\psi'\rangle = |9\rangle$, and its mean photon number will be as follows:

$$\bar{n'} =< \hat{n} >= \langle\psi'| \hat{a}^\dagger\hat{a} |\psi'\rangle = 9$$

Remind that the first state was the superposition and we only have 1/2 probability to be in the 10 photon number state. After the absorption we consider only a number state.

## Problem 3

Show that the amplitude of the fluctuation ($\Delta p_n \Delta q_n$) of number (Fock) state increases with increasing photons in the state.

We know that:

$$\hat{n} |n\rangle = n |n\rangle$$

$$\hat{a} |n\rangle = \sqrt{n} |n-1\rangle$$

$$\hat{a}^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle$$

and,

$$\langle n|m\rangle = \delta_{nm}$$

The fluctuation of the photon number , $\Delta n = \sqrt{< n^2 > - < \hat{n} >^2}$, for a Fock state takes the form:

$$< \hat{n}^2 >= \langle n| \hat{n}^2 |n\rangle = \bar{n}^2$$

and ,

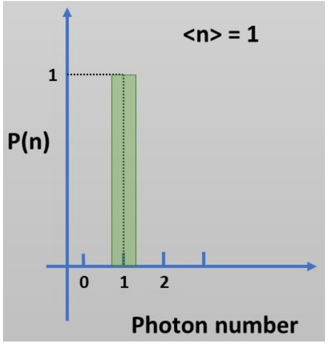
$$< \hat{n}^2 >= \bar{n}^2$$

hence,

$$(\Delta n)^2_{Fock} = 0$$

This tells that $\Delta n < \sqrt{\bar{n}} \implies$ Poissonian statistics, it has no classical electrodynamic theory (non-classical light).

The light emitters with a Fock state $n = 1$ are single-photon sources. Since they can only emit one photon at a time, the photon number probability looks as shown in the figure below:




Now, the question is to find the amplitude fluctuation.
We know that momentum and the co-ordinate operators are defined as:

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a})$$

$$\Delta p_n = \sqrt{\bar{p^2} - \bar{p}^2}$$

$$\Delta q_n = \sqrt{\bar{q^2} - \bar{q}^2}$$

Since its a Fock state,

$$\bar{p} = \langle n|\, \hat{p}\, |n\rangle = i\sqrt{\frac{\hbar\omega}{2}}\Bigg[\langle n|\, \hat{a}^\dagger\, |n\rangle - \langle n|\, \hat{a}\, |n\rangle\Bigg] = 0$$

$$\bar{q} = \langle n|\, \hat{q}\, |n\rangle = i\sqrt{\frac{\hbar}{2\omega}}\Bigg[\langle n|\, \hat{a}^\dagger\, |n\rangle + \langle n|\, \hat{a}\, |n\rangle\Bigg] = 0$$

However,

$$\bar{p^2} = \langle n|\, \hat{p^2}\, |n\rangle$$

Where,

$$\bar{p^2} = \langle n|\, \hat{p^2}\, |n\rangle = \frac{\hbar\omega}{2}[2n+1]$$

Similarly,

$$\bar{q^2} = \langle n|\, \hat{q^2}\, |n\rangle = \frac{\hbar}{2\omega}[2n+1]$$

Therefore:

$$\Delta p = \sqrt{\bar{p^2} - \bar{p}^2} = \sqrt{\frac{\hbar\omega}{2}}(2n+1)$$

$$\Delta q = \sqrt{\bar{q^2} - \bar{q}^2} = \sqrt{\frac{\hbar}{2\omega}}(2n+1)$$

Thus we have :

$$\Delta p_n \Delta q_n = \frac{\hbar}{2}(2n+1)$$

The amplitude of the fluctuation increases with increasing the number of photons in the state.

## Problem 4

Show that the uncertainity $\Delta p_\alpha \Delta q_\alpha$ of coherent state is independent of the number of photons and is equal to the vacuum state uncertainity.

A coherence state $|\alpha\rangle$ is an eigen state of annihilation operator $\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$.
Since,

$$\hat{p} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})$$

and,

$$\hat{q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a}^\dagger + \hat{a})$$

We find that:

$$\bar{p} = \langle\alpha|\,\hat{p}\,|\alpha\rangle = i\sqrt{\frac{\hbar\omega}{2}}\Bigg[\langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle - \langle\alpha|\,\hat{a}\,|\alpha\rangle\Bigg]$$

$$\bar{p} = i\sqrt{\frac{\hbar\omega}{2}}[\alpha^* - \alpha] = 2\sqrt{\frac{\hbar\omega}{2}}\,\mathrm{Im}\{\alpha\}$$

Similarly we have:

$$\bar{q} = \langle\alpha|\,\hat{q}\,|\alpha\rangle = \sqrt{\frac{\hbar}{2\omega}}\Bigg[\langle\alpha|\,\hat{a}^\dagger\,|\alpha\rangle + \langle\alpha|\,\hat{a}\,|\alpha\rangle\Bigg]$$

$$\bar{q} = \sqrt{\frac{\hbar}{2\omega}}[\alpha^* + \alpha] = 2\sqrt{\frac{\hbar}{2\omega}}\,\mathrm{Re}\{\alpha\}$$

and,

$$\bar{p^2} = -\frac{\hbar\omega}{2}\,\langle\alpha|\,\hat{a}^{\dagger 2} - \hat{a}^\dagger\hat{a} - \hat{a}\hat{a}^\dagger + \hat{a}^2\,|\alpha\rangle$$

hence we have:

$$\bar{p^2} = -\frac{\hbar\omega}{2}[(\alpha^*)^2 - 2\alpha\alpha^* - 1 + \alpha^2] = \frac{\hbar\omega}{2}[4\,\mathrm{Im}\{\alpha + 1\}]$$

Similarly,

$$\bar{q^2} = \frac{\hbar}{2\omega}\,\langle\alpha|\,\hat{a}^{\dagger 2} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger + \hat{a}^2\,|\alpha\rangle$$

$$\bar{q^2} = \frac{\hbar}{2\omega}[(\alpha^*)^2 + 2\alpha\alpha^* + 1 + \alpha^2] = \frac{\hbar\omega}{2}[4\,\mathrm{Re}\{\alpha + 1\}]$$

These leads to,

$$\Delta p_\alpha = \sqrt{\bar{p^2} - \bar{p}^2} = \sqrt{\frac{\hbar\omega}{2}}$$

and,

$$\Delta q_\alpha = \sqrt{\bar{q^2} - \bar{q}^2} = \sqrt{\frac{\hbar}{2\omega}}$$

$$\Delta p_\alpha \Delta q_\alpha = \sqrt{\frac{\hbar\omega}{2}}\sqrt{\frac{\hbar}{2\omega}} = \frac{\hbar}{2}$$

Hence the uncertainity fluctuation of coherent state isindependent of the number of photons and is equal to the vacuum state. A laser working well above its threshold can be an example.

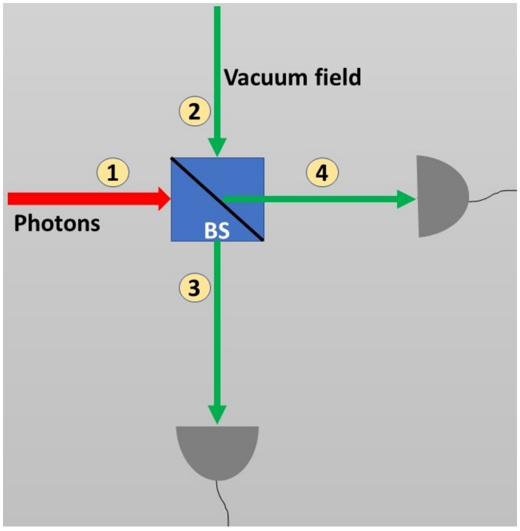
## Problem 5

Verify that the transmission of light through a beamsplitter of arbitrary loss does not change the second-order coherence.

The second order correlation of a single photon source is given by:

$$g^{(1)}(\tau) = \frac{< \hat{a}^{\dagger}(t)\hat{a}^{\dagger}(t+\tau)\hat{a}(t+\tau)\hat{a}(t) >}{< \hat{a}^{\dagger}(t)\hat{a}(t) >^2}$$

The loss can be modelled as the transmission through a lossless beamsplitter with arbitrary reflection and transmittance.
The input and the output destruction operators can be written as:




$$\hat{a}_3 = R\hat{a}_1 + T\hat{a}_2$$

$$\hat{a}_4 = T\hat{a}_1 + R\hat{a}_2$$

The electricfielf reflection and transmission coefficients R and T are complex numbers. One can write the above equation as:

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} R & T \\ T & R \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}$$

The element of the beam splitter transformation matrix M is unitary. Which imples thta,

$$M^{-1}M = M^{+}M = 1$$

or

$$M^{-1} = M^{+}$$

Hence ,

$$|R|^2 + |T|^2 = 1$$

and,

$$R^{*}T + T * R = 0$$

One can write for example:

$$g^{(44)}(0) = \frac{< \hat{a_4}^\dagger \hat{a_4}^\dagger \hat{a_4}\hat{a_4} >}{< \hat{a_4}^\dagger \hat{a_4} >^2}$$

$$g^{(44)}(0) = |T|^4 \frac{< \hat{a}^\dagger(t)\hat{a}^\dagger(1)\hat{a}(1)\hat{a}(t) >}{|\tau|^2|\tau|^2 < \hat{a}^\dagger(t)\hat{a}(t) >^2} = g^{(33)}(0) = g^{(11)}(0)$$

Hence second order coherence of the transmitted and reflected fields are equal to that of the incident light. Which implies that the $g^{(2)}$ can be measured accurately in an optical system that has low transmission or low extraction efficiency.

## Problem 6

Consider a single photon source that emits photons at a rate of $10^7 s^{-1}$ and has a $g^{(2)}(0) = 0.25$.

- Determine the probability of the source emitting one or more photons over a time interval of 1 ns.
- Determine the two photon probability for the same interval.

We know that:

$$g^{(1)}(\tau) = \frac{< \hat{a}^\dagger(t)\hat{a}^\dagger(t+\tau)\hat{a}(t+\tau)\hat{a}(t) >}{< \hat{a}^\dagger(t)\hat{a}(t) >^2}$$

$$g^{(2)}(0) = \frac{< \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} >}{< \hat{a}^\dagger\hat{a} >^2} = \frac{< \hat{n}(\hat{n}-1) >}{< \hat{n} >^2} = \frac{\mathrm{Tr}\{\hat{\rho}\hat{n}(n \hat{-} 1)\}}{\Big(\mathrm{Tr}\{\hat{\rho}\hat{n}\}\Big)^2}$$

Only the diagonal terms of the density matrix contribute to $g^{(2)}(0)$. Hence,

$$g^{(2)}(0) = \frac{\sum_{n=0}^{\infty} n(n-1)P(n)}{\Big[\sum_{n=0}^{\infty} nP(n)\Big]^2} = \frac{2P(2) + 6P(3) + ..........}{\bar{n}^2}$$

Where, $\bar{n}$ is the mean photon number. In the case where $P(1) >> P(2) >> P(n > 2)$, which is true for most single photon sources.

$$g^{(2)}(0) = \frac{2P(2)}{\bar{n}^2} = \frac{2P(2)}{[P(1)]^2}$$

Note that $g^{(2)}(0)$ does not directly reflect the two ; one photon ratio $(P(2)/P(1))$, but it is a relevant quantity to show the source quality.
Since the source under consideration is a single photon source that emits photons at a ratio of $R = 10^7 s^{-1}$ and has $g^{(2)}(0) = 0.25$, the probability of the source emitting one or more photon over a time interval of 1 ns can be find as:

$$P(n > 0) = 10^{-9}s \times 10^7 s^{-1} = 10^{-2}$$

With this low probability, we can approximate $P(1) \simeq P(n > 0)$ and $P(2) \simeq \frac{1}{2}g^{(2)}(0)[P(1)]^2 \simeq \frac{1}{2} \times 0.25 \times 10^{-4} = 1.25 \times 10^{-5}$.
The two to one photon ratio is :

$$\frac{P(2)}{P(1)} \simeq \frac{1.25 \times 10^{-5}}{10^{-2}} \simeq 1.25 \times 10^{-3}$$