Problem 1

- In the fully quantized model of a two-level atom interacting with a quantized field within the rotating wave approximation (think about the Jaynes–Cummings model), we have obtained the exact resonance solution for the initial state where the atom is excited and where the field is in a number state $|n\rangle$. Recall that the Rabi oscillations of the atomic inversion are periodic for this case. Use that solution to obtain the expectation value of the atomic dipole moment operator? Comment on the result.

- Obtain the expectation value of the atomic dipole moment as given by the Jaynes–Cummings model in the case where the field is initially in a coherent state. How does the result compare with the two previous cases? You should make a plot of the expectation value of the dipole moment as a function of time.

The atom is initially in the excited state $|e\rangle$ and the field is intially in the number state $|n\rangle$. The initial state of the atom field system is then:

$$|i\rangle = |e\rangle\,|n\rangle$$

withe energy $E_i = \frac{1}{2}\hbar\omega + n\hbar\omega$. This state is coupled to state $|f\rangle = |g\rangle\,|n+1\rangle$ with energy $E_f = -\frac{1}{2}\hbar\omega + (n+1)\hbar\omega$.
Remind that $E_i = E_f$. The state vector at any time time t is given by:

$$|\psi(t)\rangle = C_i(t)\,|i\rangle + C_f(t)\,|f\rangle$$

where $C_i(0) = 1$ and $C_f(0) = 0$. From the interaction picture we find the coefficients:

$$\dot{C}_i = -i\lambda\sqrt{n+1}\;C_f$$

$$\dot{C}_f = -i\lambda\sqrt{n+1}\;C_i$$

By eliminating C_f we obtain:

$$\ddot{C}_i + \lambda^2(n+1)\;C_i = 0$$

The solution can be (using the initial conditions):

$$\boxed{C_i(t) = \cos\left(\lambda t\sqrt{n+1}\right)}$$

$$\boxed{C_f(t) = -i\sin\left(\lambda t\sqrt{n+1}\right)}$$

Hence:

$$|\psi(t)\rangle = \cos\left(\lambda t\sqrt{n+1}\right)|e\rangle\,|n\rangle - i\sin\left(\lambda t\sqrt{n+1}\right)|g\rangle\,|n+1\rangle$$

Now let us remember the equation:

$$\hat{d}\,|\psi(t)\rangle = ?$$

$$< \hat{d} >=?$$

$$\hat{d} = d\left|g\right\rangle\left\langle e\right| + d^*\left|e\right\rangle\left\langle g\right|$$

$$\hat{d} = d\hat{\sigma}_- + d^*\hat{\sigma}_+ = d(\hat{\sigma}_+ + \hat{\sigma}_-)$$

Therefore,

$$\hat{d}\left|\psi(t)\right\rangle = \left[\cos\left(\lambda t\sqrt{n+1}\right)\left|g\right\rangle\left|n\right\rangle - i\sin\left(\lambda t\sqrt{n+1}\right)\left|e\right\rangle\left|n+1\right\rangle\right]d$$

and,

$$\boxed{< \hat{d} >= \left\langle\psi(t)\right|\hat{d}\left|\psi(t)\right\rangle = 0}$$

This is due to the entanglement between the photon number state and the atom.

When the initial state of the photon is:

$$\boxed{\left|\alpha\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}}$$

and the atom initial state is: $\left|\psi_a\right\rangle = \left|e\right\rangle$.

Hence the initial state can be written as:

$$\left|\psi\right\rangle_i = \left|\alpha\right\rangle\left|e\right\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left|n\right\rangle\left|e\right\rangle$$

At a later time t, the state is transformed into:

$$\left|\psi(t)\right\rangle = \sum_{n=0}^{\infty}\left(C_{e,n}(t)\left|n\right\rangle\left|e\right\rangle + C_{g,n}(t)\left|n+1\right\rangle\left|g\right\rangle\right)$$

Using Schrödinger interaction picture,

$$\boxed{i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = \hat{H}_I\left|\psi(t)\right\rangle}$$

$$i\hbar\frac{d}{dt}\left|\psi(t)\right\rangle = i\hbar\sum\left(C_{e,n}(t)\left|n\right\rangle\left|e\right\rangle + C_{g,n}(t)\left|n+1\right\rangle\left|g\right\rangle\right)$$

$$\hat{H}_I\left|\psi(t)\right\rangle = \hbar\lambda\sum\left(\sqrt{n+1}\ C_{e,n}(t)\left|n+1\right\rangle\left|g\right\rangle + \sqrt{n+1}\ C_{g,n}(t)\left|n\right\rangle\left|e\right\rangle\right)$$

Hence $\dot{C}_{e,n}(t) + \lambda^2(n+1)C_{e,n}(t) = 0$.
The solution take the form:

$$C_{e,n}(t) = A_n\cos\left(\lambda\sqrt{n+1}\ t\right) + B_n\sin\left(\lambda\sqrt{n+1}\ t\right)$$

Since,

$$C_{g,n}(t) = \frac{i}{\lambda\sqrt{n+1}}\dot{C}_{e,n}(t) = -A_n \sin\big(\lambda\sqrt{n+1}\ t\big) + B_n \cos\big(\lambda\sqrt{n+1}\ t\big)$$

Using the initial conditions,

$$A_n = C_{e,n}(0) = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}}$$

and,

$$B_n = 0$$

These leads to:

$$\boxed{C_{e,n}(t) = e^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \cos\big(\lambda\sqrt{n+1}\ t\big)}$$

$$\boxed{C_{g,n}(t) = -ie^{-\frac{|\alpha|^2}{2}} \frac{\alpha^n}{\sqrt{n!}} \sin\big(\lambda\sqrt{n+1}\ t\big)}$$

The state at any time t can be written as:

$$|\psi(t)\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left[\cos\left(\lambda t\sqrt{n+1}\right)|n\rangle|e\rangle - i\sin\left(\lambda t\sqrt{n+1}\right)\ |n+1\rangle|g\rangle\right]$$

The question is to find $<\hat{d}>= \langle\psi(t)|\,\hat{d}\,|\psi(t)\rangle$, but,

$$\hat{d}\,|\psi(t)\rangle = de^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left[\cos\left(\lambda t\sqrt{n+1}\right)|n\rangle|g\rangle - i\sin\left(\lambda t\sqrt{n+1}\right)\ |n+1\rangle|e\rangle\right]$$

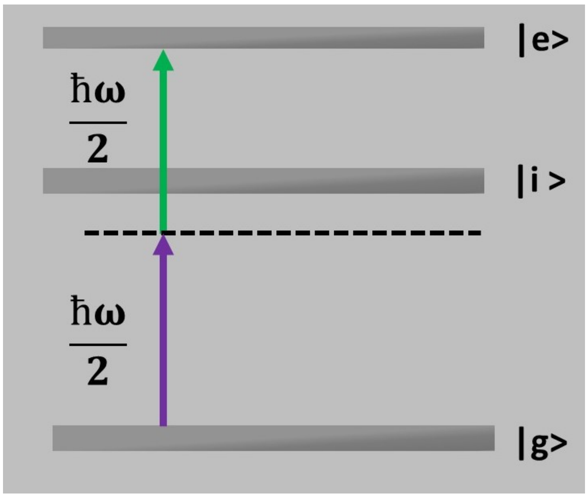
$$\boxed{\langle\psi(t)|\,\hat{d}\,|\psi(t)\rangle = -2\,\mathrm{Im}\{\alpha\}\ d\ e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!\ \sqrt{n+1}} \left[\cos\left(\lambda\sqrt{n+2}\ t\right)\sin\left(\lambda\sqrt{n+1}\ t\right)\right]}$$

Problem 2

A resonant two-photon extension of the Jaynes–Cummings model is described by the effective Hamiltonian:

$$\hat{H}_{eff} = \hbar\eta(\hat{a}^2\hat{\sigma}_+ + \hat{a}^{2\dagger}\hat{\sigma}_-)$$

where, for the sake of simplicity, a small Stark shift term has been ignored. This Hamiltonian represents two-photon absorption and emission between atomic levels of like parity. The process is represented by Fig.below, where the broken line represents a virtual intermediate state of opposite parity.



- **Obtain the dressed states for this system.**

  The resonant two-photon extension of the Jaynes–Cummings model is described by the effective Hamiltonian:

$$\hat{H}_{eff} = \hbar\eta(\hat{a}^2\hat{\sigma}_+ + \hat{a}^{2\dagger}\hat{\sigma}_-)$$

Now let us define our states:

$$|i\rangle = |e\rangle\,|n\rangle$$

$$|f\rangle = |g\rangle\,|n+2\rangle$$

To obtain the dressed states and energy eigenvalues, matrix mechanics will be better. The matrix elements are:

$$\langle i|\,\hat{H}_{eff}\,|i\rangle = 0$$

$$\langle f|\,\hat{H}_{eff}\,|i\rangle = \hbar\eta\sqrt{(n+2)(n+1)}$$

$$\langle f|\,\hat{H}_{eff}\,|f\rangle = 0$$

$$\langle i|\,\hat{H}_{eff}\,|f\rangle = \hbar\eta\sqrt{(n+2)(n+1)}$$

Hence:

$$H^{(n)} = \begin{pmatrix} 0 & \hbar\eta\sqrt{(n+2)(n+1)} \\ \hbar\eta\sqrt{(n+2)(n+1)} & 0 \end{pmatrix}$$

The corresponding eigenvalue (energy) and the eigenstates (dressed states) takes the form:

$$\boxed{E_{\pm} = \pm\hbar\eta\sqrt{(n+2)(n+1)}}$$

$$\boxed{|n,+\rangle = \frac{1}{\sqrt{2}}(|i\rangle + |f\rangle)}$$

$$\boxed{|n,-\rangle = \frac{1}{\sqrt{2}}(|i\rangle - |f\rangle)}$$

- **Obtain the atomic inversion for this model assuming the atom initially in the ground state and that the field is initially in a number state. Repeat for a coherent state. Comment on the nature of the collapse and revival phenomena for these states.**

The initial state of the atom field is:

$$|\psi_{a,f}(0)\rangle = |g\rangle \;\; |n+2\rangle$$

$$|\psi_{a,f}(0)\rangle = \frac{1}{\sqrt{2}}(|n,+\rangle - |n,-\rangle)$$

At a later time the state is transformed into:

$$|\psi_{a,f}(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} \, |\psi_{a,f}(0)\rangle$$

$$|\psi_{a,f}(t)\rangle = e^{\frac{-i\hat{H}t}{\hbar}} \frac{1}{\sqrt{2}}[|n,+\rangle - |n,-\rangle]$$

$$|\psi_{a,f}(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{\frac{-iE_{+}t}{\hbar}} \, |n,+\rangle - e^{\frac{-iE_{-}t}{\hbar}} \;\; |n,-\rangle\right)$$

$$|\psi_{a,f}(t)\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\eta\sqrt{(n+2)(n+1)}\;t} \, |n,+\rangle - e^{i\eta\sqrt{(n+2)(n+1)}\;t} \;\; |n,-\rangle\right)$$

$$|\psi_{a,f}(t)\rangle = i\sin\left(\eta\sqrt{(n+2)(n+1)}\;t\right)|i\rangle + \cos\left(\eta\sqrt{(n+2)(n+1)}\;t\right)$$

The atomic inversion is given by:

$$W(t) = \langle\psi_{a,f}(t)|\,\hat{\sigma}_3\,|\psi_{a,f}(t)\rangle = P_i(t) - P_f(t)$$

Hence:

$$\boxed{W(t) = \sin^2\left(\eta\sqrt{(n+2)(n+1)}\;t\right) - \cos^2\left(\eta\sqrt{(n+2)(n+1)}\;t\right)}$$

$$\boxed{W(t) = -\cos\left(2\;\eta\sqrt{(n+2)(n+1)}\;\;t\right)}$$

When the field is initially in a coherent state:

$$\begin{aligned}
|\psi_{a,f}(0)\rangle &= |g\rangle |\alpha\rangle \\
&= \sum_{n=0}^{\infty} C_n |g\rangle |n\rangle \\
&= |g\rangle (C_0 |0\rangle + C_1 |1\rangle) + \sum_{n=0}^{\infty} C_{n+2} \ |g\rangle |n+2\rangle \\
&= |g\rangle (C_0 |0\rangle + C_1 |1\rangle) + \sum_{n=0}^{\infty} C_{n+2} \ |f_n\rangle \\
&= |g\rangle (C_0 |0\rangle + C_1 |1\rangle) + \sum_{n=0}^{\infty} C_{n+2} \ \frac{1}{\sqrt{2}}(|n,+\rangle - |n,-\rangle)
\end{aligned}$$

This allows us to easily calculate the state vector at a later time t.

$$\begin{aligned}
|\psi_{a,f}(t)\rangle &= e^{\frac{-i\hat{H}t}{\hbar}} \ |\psi_{a,f}(0)\rangle \\
&= |g\rangle (C_0 |0\rangle + C_1 |1\rangle) + \sum_{n=0}^{\infty} C_{n+2} \frac{1}{\sqrt{2}} \Big(e^{-iE_+ t} |n,+\rangle - e^{-iE_- t} |n,-\rangle \Big) \\
&= |g\rangle (C_0 |0\rangle + C_1 |1\rangle) + \sum_{n=0}^{\infty} C_{n+2} \ \ i \ \sin\Big(i \ \sin\Big(\eta\sqrt{(n+2)(n+1)} \ t\Big)\Big) |i\rangle + \\
&\cos\Big(\eta\sqrt{(n+2)(n+1)} \ t\Big) |f\rangle \\
&= |g\rangle \Bigg[C_o |0\rangle + C_1 |1\rangle + i \ \sum_{n=0}^{\infty} C_{n+2} \ \sin\Big(\eta\sqrt{(n+2)(n+1)} \ t\Big) |n+2\rangle \\
&+ \ |e\rangle \sum_{n=0}^{\infty} C_{n+2} \ \cos\Big(\eta\sqrt{(n+2)(n+1)} \ t\Big) |n\rangle \Bigg]
\end{aligned}$$

Hence:

$$W(t) = \langle\psi_{a,f}(t)| \hat{\sigma}_3 |\psi_{a,f}(t)\rangle$$

takes the form:

$$W(t) = \Bigg[|C_0|^2 + |C_1|^2 + \sum_{n=0}^{\infty} |C_{n+2}|^2 \ \sin^2(\eta\sqrt{(n+2)(n+1)}) \ t\Bigg] - \Bigg[\sum_{n=0}^{\infty} |C_{n+2}|^2 \ \cos^2(\eta\sqrt{(n+2)(n+1)}) \ t\Bigg]$$

$$W(t) = |C_0|^2 + |C_1|^2 - \sum_{n=0}^{\infty} |C_{n+2}|^2 \ \cos\Big(2\eta\sqrt{(n+2)(n+1)}t\Big)$$

Problem 3

Consider a two-level system initially in an excited state is placed in a cavity of quality factor $Q = \frac{\omega_0}{\kappa}$. The cavity supports only single mode. The interaction Hamiltonian is given by,

$$\hat{H}_1 = \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)$$

and the master equation for the evolution of the density operator is given by,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_1, \hat{\rho}] - \frac{\kappa}{2}(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}) + \kappa\hat{a}\hat{\rho}\hat{a}^\dagger$$

- Solve the master equation to determine the decay rate of the excited state.
- What will be the excited state decay rate if the quality factor of the cavity is very high such that $\frac{\omega_0}{Q} < 2\Omega_0$.
- What will be the excited state decay rate in the case of strong cavity damping $\frac{\omega_0}{Q} > 2\Omega_0$. Discuss the results.

We know that the interaction Hamiltonian is given by:

$$\boxed{\hat{H}_1 = \hbar\lambda(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)}$$

and the master equation for the evolution of the density operator is given by,

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_1, \hat{\rho}] - \frac{\kappa}{2}(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}) + \kappa\hat{a}\hat{\rho}\hat{a}^\dagger$$

where $\kappa = \frac{\omega_0}{Q}$, where Q is a characteristic quality factor parameter of the cavity describing the rate of loss. Because of the loss we have 3 states.

$$\boxed{|1\rangle = |e\rangle\,|0\rangle \implies \text{the atom is in the excited state.}}$$

$$\boxed{|2\rangle = |g\rangle\,|1\rangle \implies \text{the atom is in the ground state by emitting a photon.}}$$

The energy of $|1\rangle$ and $|2\rangle$ are the same.

$$\boxed{|3\rangle = |g\rangle\,|0\rangle \implies \text{the atom is in the ground state but the photon is lost.}}$$

With the states $|i\rangle$, i = 1,2,3, we can write:

$$\rho_{ij} = \langle i|\,\hat{\rho}\,|i\rangle$$

and the master equation will take the following form:

$$\frac{\mathrm{d}\rho_{11}}{\mathrm{d}t} = \frac{i}{2}\,\Omega_0(\rho_{12} - \rho_{21}) = \frac{i}{2}\,\Omega_0 V$$

$$\frac{\mathrm{d}\rho_{22}}{\mathrm{d}t} = -\frac{\omega_0}{Q}\,\rho_{22} - \frac{i}{2}\,\Omega_0 V$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -i\Omega_0 W - \frac{1}{2}(\frac{\omega_0}{Q})V$$

$$\frac{\mathrm{d}\rho_{33}}{\mathrm{d}t} = \frac{\omega_0}{Q}\rho_{22}$$

where $V = \rho_{12} - \rho_{21}$ and $W = \rho_{11} - \rho_{22}$.
We can use matrix representation to solve the problem.

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\rho_{11}\\ \rho_{22}\\ V\end{pmatrix} = \begin{pmatrix} 0 & 0 & i\frac{\Omega_0}{2} \\ 0 & -\frac{\omega_0}{Q} & -i\frac{\Omega_0}{2} \\ i\Omega_0 & -i\Omega_0 & -\frac{\omega_0}{2Q}\end{pmatrix}\begin{pmatrix}\rho_{11}\\ \rho_{22}\\ V\end{pmatrix}$$

The initial conditions are $\rho_{11}(0) = 1$, $\rho_{22}(0) = 0$, and $\rho_{12}(0) = 0$, $\rho_{33}(0) = 0$.

The eigenvalue problem of the above matrix:

$$\left(\Delta + \frac{\omega_0}{2Q}\right)\left(\Delta^2 + \frac{\omega_0}{Q}\Delta + \Omega_0^2\right) = 0$$

and the solutions are:

$$\Delta_0 = -\frac{\omega_0}{2Q}$$

$$\Delta_\pm = -\frac{\omega_0}{2Q} \pm \frac{\omega_0}{2Q}\left(1 - \frac{4\Omega_0^2 Q^2}{\omega_0^2}\right)^{1/2}$$

When the cavity decay rate is weak such that $\boxed{\frac{\omega_0}{Q} << 2\Omega_0}$ the eigenvalues $\Delta_\pm$ will be complex. This leads to damped oscillations of frequency Ω_0 in the probability of finding the dipole (atom) in the excited states, $P_e(t) = \rho_{11}(t)$.

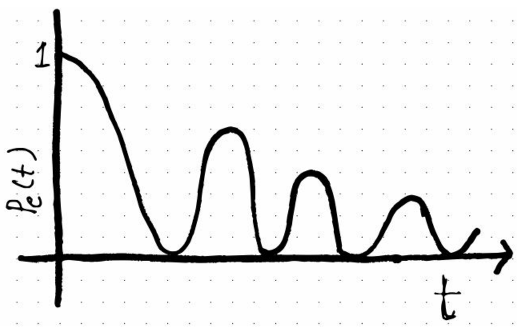
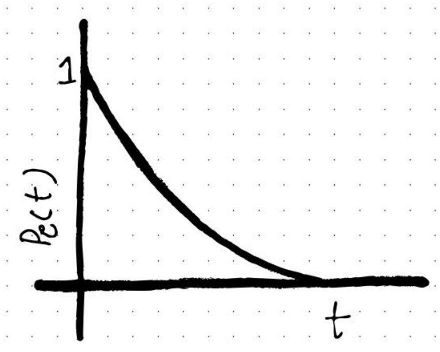


Figure 1: Damped excited state probability for an atom in a high-Q cavity

The oscillations reflect the spontaneous emission is reversibile for weak-field decay.
When there is strong cavity damping $\boxed{\frac{\omega_0}{Q} >> 2\Omega_0}$, the eigenvalues $\Delta_\pm$ are real and this leads to irreversible spontaneous emission.
The largest the time constant, thus the smallest eigenvalues in this case is Δ_+.

$$\Delta_+ \simeq \frac{-\Omega_0^2 Q}{\omega_0} = -\frac{4d^2 Q}{\hbar\epsilon_0 V}$$

remember that $\Omega_0 = 2\lambda$, $\lambda = dg/\hbar$ and V is the effective mode volume.



$$g = -\left(\frac{\hbar\omega}{\epsilon_0 V}\right)^{1/2}$$

Hence the rate of irreversible decay becomes:

$$\boxed{\Gamma_{cav} = \frac{4d^2 Q}{\hbar\epsilon_0 V}}$$

In the experiment one can manipulate the quality factor and mode volume of the cavity to control the decay rate. Application: Single-photon sources, low-threshold lasers and so on.