Qubits and Quantum gates

1. **Show that the Pauli matrix σ_x (it is also called X gate) is the quantum analog of the NOT gate.**

   Classical NOT gate logic table is the following:

   Table 1: NOT gate logic table

Input	**0**	**1**
Output	**1**	**0**

   But quantum gate X is defined by the unitary matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and its action is given by:

   $$|0\rangle - \boxed{X} - |1\rangle$$

   $$|1\rangle - \boxed{X} - |0\rangle$$

   or

   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \implies \hat{X}|0\rangle = |1\rangle$$

   and

   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies \hat{X}|1\rangle = |0\rangle$$

   Since $X = \sigma_x$, the Pauli matrix is also considered to be the quantum analog of NOT gate.

2. **Consider a two-level system (qubit of the NV color center) defined by $\alpha|0\rangle + \beta|1\rangle$, show that the operation of X gate on the qubit exchange the probability of finding the qubit in the $|0\rangle$ and $|1\rangle$ state.**

   The qubit state is defined by:

   $$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

   The operation of X gate on this state is given as:

   $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\big[\alpha|0\rangle + \beta|1\rangle\big] \to \beta|0\rangle + \alpha|1\rangle$$

   Hence,

   $$\alpha|0\rangle + \beta|1\rangle - \boxed{X} - \beta|0\rangle + \alpha|1\rangle$$

Therefore the operation leads to a state with $|\beta|^2$ probability to find the qubit in $|0\rangle$ state and $|\alpha|^2$ probability to find the qubit in $|1\rangle$ state.

3. **Show that the Z gate does not change the state of the qubit.**

   The Z gate is defined by the unitary matrix:

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

   and its action is:

$$|0\rangle - \boxed{Z} - |0\rangle = \hat{Z}\,|0\rangle = |0\rangle$$

$$|1\rangle - \boxed{Z} - [-\,|1\rangle] = \hat{Z}\,|1\rangle = -\,|1\rangle$$

4. **Show that the H gate brings the qubit state from $|0\rangle$ (or $|1\rangle$) state to superposition states. What is the probability of finding $|0\rangle$ and $|1\rangle$ states after the application of H gate.**

The H gate is defined by unitary matrix:

$$\hat{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and its action is:

$$|0\rangle - \boxed{H} - \left[\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\right]$$

$$|1\rangle - \boxed{H} - \left[\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right]$$

When we measure, we obtain 0 or 1, each with 50% probability. This help to have a circuit that generates perfectly uniform random bits.

Quantum states readout

We know that the no-cloning theorem forbids us from copying arbitrary quantum states,

1. **Show that if there exists a unitary U taking $|\psi\rangle \to |\phi\rangle$, there must exist another unitary V independent of $|\psi\rangle$ and $|\phi\rangle$. In another words, no information is lost when applying a unitary to a quantum state.**

   Suppose there exists a unitary U mapping $|\psi\rangle \to |\phi\rangle$. Since U is unitary, $U^{-1} = U^*$ is unitary. So the operator $V = U^*$ is a unitary operator which maps $|\phi\rangle \to |\psi\rangle$.

2. **Suppose Alice holds a qubit in the state $|\psi\rangle = a|0\rangle + b|1\rangle$ and wants Bob to have that state as well. Why doesn't the following work? Alice measures her qubit in some basis of her choice, then prepares another qubit in the state she obtains and sends that to Bob. (Please answer without making explicit use of the no-cloning theorem.)**

   We now introduce a scheme by which Alice can prepare $|\psi\rangle$ on Bob's side without sending him her qubit in fact, without sending any quantum information at all!, provided they share a Bell state. To be precise, the initial setup is as follows: Alice holds $|\psi\rangle_s$ and Alice and Bob each have a qubit of:

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A |0\rangle_B + |1\rangle_A |1\rangle_B) \tag{1}$$

   So that the joint state of all three qubits is:

$$|\psi\rangle_{SAB} = |\psi\rangle_S \otimes |\phi^+\rangle_{AB} = \Big(a|0\rangle_S + b|1\rangle_S\Big) \otimes \frac{1}{\sqrt{2}}\Big(|00\rangle_{AB} + |11\rangle_{AB}\Big) \tag{2}$$

   If Alice measures $|\psi\rangle$ in a basis of her choice, then $|\psi\rangle$ will collapse to some eigenstate of the basis . So, if she reproduces this state and sends it to Bob, Bob will have a state that is different than the original state unless $|\psi\rangle$ happens to be an element of the basis in which Alice is measuring.

3. **As with a single qubit, any state of a two-qubit system can be written in terms of an orthonormal basis, and also measured in such a basis. One example is the computational basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. Find a state $|\psi^-\rangle$ that, together with the following three states:**

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB})$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB})$$

   $|\psi^-\rangle$ forms an orthonormal basis of the two qubit space. This basis is called the Bell states.

   Let

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})$$

   Then,

$$\langle \psi^{\pm} | \psi^{\pm} \rangle = \frac{1}{2}(1+1) = 1$$

$$\langle \psi^{\mp} | \psi^{\pm} \rangle = \frac{1}{2}(1-1) = 0$$

$$\langle \psi^{+} | \phi^{\pm} \rangle = 0$$

$$\langle \phi^{\pm} | \phi^{\pm} \rangle = \frac{1}{2}(1+1) = 1$$

$$\langle \phi^{\mp} | \phi^{\pm} \rangle = \frac{1}{2}(1-1) = 0$$

$$\langle \phi^{+} | \psi^{\pm} \rangle = 0$$

4. **Rewrite the joint state (2) as a linear combination of the form $\sum_{i=1}^{4} = |\alpha_i\rangle_{SA} |\beta_i\rangle_B$, where $|\alpha\rangle$ ranges over the four possible Bell states on Alice's two qubits S and A, and $|\beta\rangle$ is a single qubit state on Bob's qubit.**

   Note:

$$\begin{aligned} |\psi\rangle_{SAB} &= \Big(a\,|0\rangle_S + b\,|1\rangle_S\Big) \otimes \frac{1}{\sqrt{2}}\Big(|00\rangle_{AB} + |11\rangle_{AB}\Big) \\ &= \frac{1}{\sqrt{2}}\Big(a\,|000\rangle_{SAB} + b\,|100\rangle_{SAB} + a\,|011\rangle_{SAB} + b\,|111\rangle_{SAB}\Big) \end{aligned}$$

   and

$$\frac{1}{2}\Big(|\phi^+\rangle + |\phi^-\rangle\Big) = \frac{1}{\sqrt{2}}|00\rangle$$

$$\frac{1}{2}\Big(|\psi^+\rangle + |\psi^-\rangle\Big) = \frac{1}{\sqrt{2}}|01\rangle$$

$$\frac{1}{2}\Big(|\phi^+\rangle - |\phi^-\rangle\Big) = \frac{1}{\sqrt{2}}|11\rangle$$

$$\frac{1}{2}\Big(|\psi^+\rangle - |\psi^-\rangle\Big) = \frac{1}{\sqrt{2}}|10\rangle$$

   Then we have,

$$\begin{aligned} |\psi_{AB}\rangle &= \frac{1}{2}\Bigg(a\Big(|\phi^+\rangle + |\phi^-\rangle\Big)|0\rangle + b\Big(|\psi^+\rangle - |\psi^-\rangle\Big)|0\rangle + a\Big(|\psi^+\rangle + |\psi^-\rangle\Big)|1\rangle + b\Big(|\phi^+\rangle - |\phi^-\rangle\Big)|1\rangle\Bigg) \\ &= \frac{1}{2}\Bigg(|\phi^+\rangle\Big(a\,|0\rangle + b\,|1\rangle\Big) + |\phi^-\rangle\Big(a\,|0\rangle - b\,|1\rangle\Big) + |\psi^+\rangle\Big(b\,|0\rangle + a\,|1\rangle\Big) + |\psi^-\rangle\Big(-b\,|0\rangle + a\,|1\rangle\Big)\Bigg) \end{aligned}$$

5. **Suppose Alice measures her two qubits SA in the Bell basis and sends the results to Bob. Show that for each of the four possible outcomes, Bob can use this (classical!) information to determine a unitary, independent of $|\psi\rangle_S$, on his qubit that will map it, in all cases, to the original state $|\psi\rangle_B$ that Alice had.**

   From part (4), the possible outcomes are , $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle$ and $|\psi^-\rangle$ in which case Bob's qubit is $a|0\rangle + b|1\rangle, a|0\rangle - b|1\rangle, b|0\rangle + a|1\rangle, -b|0\rangle + a|1\rangle$ resepectively.

   If Alice measures $|\phi^+\rangle$ then Bob should apply the identity.

   If Alice measures $|\phi^-\rangle$ then Bob should apply $Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

   If Alice measures $|\psi^+\rangle$ then Bob should apply $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

   If Alice measures $|\psi^-\rangle$ then Bob should apply $ZX = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.